Matrices and Determinants

Level - 3	Daily Tutorial Sheet-15

168. $A^{p+1} = (B+C)^{p+1}$

We can expand $(B+C)^{p+1}$ like binomial expansion as $BC = CB$.

$\therefore \quad (B+C)^{p+1} = {}^{p+1}C_0 B^{p+1} + {}^{p+1}C_1 B^{p} C + \ldots + {}^{p+1}C_{p+1} C^{p+1}$

$= {}^{p+1}C_0 B^{p+1} + {}^{p+1}C_1 B^{p} C + O + O + \ldots + O \qquad (\because C^2 = O \Rightarrow C^3 = C^4 = \ldots = O)$

$= B^{p+1} + (p+1) B^p C = B^p \left[B + (p+1) C \right]$

169. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$

$\Rightarrow \quad f(D) = a_0 I + a_1 D + a_2 D^2 + \ldots + a_n D^n$

$a_0 \times \text{diag}(1, 1, \ldots, 1)$

$+ a_1 \times \text{diag}(d_1, d_2, \ldots., d_n)$

$+ a_2 \times \text{diag}\left(d_1^2, d_2^2, \ldots, d_n^2\right)$

$+ ..$

$\vdots$

$+ a_n \times \text{diag}\left(d_1^n, d_2^n, \ldots, d_n^n\right)$

$= \text{diag}(a_0 + a_1 d_1 + a_2 d_1^2 + \ldots a_n d_1^n,$

$a_0 + a_1 d_2 + a_2 d_2^2 + \ldots + a_n d_2^n,$

$a_0 + a_1 d_3 + a_2 d_3^2 + \ldots + a_n d_3^n$

$\vdots$

$a_0 + a_1 d_n + a_2 d_n^2 + \cdots \ldots a_n d_n^n) = \text{diag}\left(f(d_1), f(d_2), \ldots f(d_n)\right)$

170. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square root of the matrix.

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then $A^2 = I \quad \Rightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Or $\quad \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & cb + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Or $\quad a^2 + bc = 1 \qquad (1)$

$ab + bd = 0 \qquad (2)$

$ac + cd = 0 \qquad (3)$

$cb + d^2 = 1$ (4)

If $a + d = 0$, the above four equations hold simultaneously if $d = -a$ and $a^2 + bc = 1$. Hence, one possible square root of I is $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$

Where α, β, γ are the three numbers related by the condition $\alpha^2 + \beta\gamma = 1$.

If $a + d \neq 0$, then above four equations hold simultaneously

If $b = 0, c = 0, a = 1, d = 1$ or if $b = 0, c = 0$, $a = -1, d = -1$.

Hence, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ i.e., $\pm I$ are other possible square roots of I.

171. First, we will show that $I - S$ is nonsingular. The equality $|I - S| = 0$ implies that I is a characteristic root of the matrix S, but this is not possible, for a real skew-symmetric matrix can have zero or purely imaginary numbers as its characteristic roots. Thus. $|I - S| \neq 0$, i.e., $I - S$ is nonsingular. We have

$$A^T = \left[(I - S)^{-1}\right]^T (I + S)^T = \left[(I - S)^T\right]^{-1} (I + S)^T$$

But $(I - S)^T = I^T - S^T = I + S$ and $(I + S)^T = I^T + S^T = I - S$

$\therefore$ $A^T = (I + S)^{-1}(I - S)$

$\therefore$ $A^T A = (I - S)^{-1}(1 - S)(I + S)(I - S)^{-1} = (I - S)^{-1}(I + S)(I - S)(I - S)^{-1} = I$

Thus, A is orthogonal.

172. $(xf)' = xf' + f$ and $(x^2 f)'' = [2xf + x^2 f']' = 2f + 4xf' + x^2 f''$

$$\Rightarrow \quad \Delta = \begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ 2f + 4xf' + x^2 f'' & 2g + 4xg' + x^2 g'' & 2h + 4xh' + x^2 h'' \end{vmatrix}$$

$R_2 \to R_2 - R_1$ and then $R_3 \to R_3 - 4R_2 - 2R_1$

$$\Rightarrow \quad \Delta = \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix}$$

Taking x common from R_2 and multiplying with R_3, we have $\Delta = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix}$

$$\Rightarrow \quad \frac{d\Delta}{dx} = \begin{vmatrix} f' & g' & h' \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

173. Since α is a repeated root of the quadratic equation $f(x)=0$, $f(x)$ can be written as $f(x)=k(x-a)^2$, where k is some nonzero constant.

Let $g(x)=\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ $g(x)$ is divisible by $f(x)$ if it is divisible by $(x-\alpha)^2$, i.e., $g(\alpha)=0$ and $g'(\alpha)=0$.

As $A(x)$, $B(x)$, and $C(x)$ are polynomials of degrees 3, 4 and 5, respectively deg. $g(x)\ge 2$.

Now, $g(\alpha)=\begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}=0$ (R_1 and R_2 are identical)

Also $g'(x)=\begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ $\therefore$ $g'(\alpha)=\begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}=0$ (R_1 and R_3 are identical))

This implies that $f(x)$ divides $g(x)$.