

Matrices and Determinants

Level - 3 Daily Tutorial Sheet-15

168.
$$A^{p+1} = (B+C)^{p+1}$$

We can expand $(B+C)^{p+1}$ like binomial expansion as BC=CB.

$$(B+C)^{p+1} = {}^{p+1}C_0B^{p+1} + {}^{p+1}C_1B^pC + \dots + {}^{p+1}C_{p+1}C^{p+1}$$

$$= {}^{p+1}C_0B^{p+1} + {}^{p+1}C_1B^pC + O + O + \dots + O \qquad \qquad (\because C^2 = O \Rightarrow C^3 = C^4 = \dots = O)$$

$$= B^{p+1} + (p+1)B^pC = B^p \lceil B + (p+1)C \rceil$$

169. Let
$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$$

$$\Rightarrow f(D) = a_0I + a_1D + a_2D^2 + ... + a_nD^n$$

$$a_0 \times \operatorname{diag}(1,1,...,1)$$

$$+a_1 \times \operatorname{diag}(d_1,d_2,...,d_n)$$

$$+a_2 \times \operatorname{diag}(d_1^2,d_2^2,...,d_n^2)$$
+..
:
$$+a_n \times \operatorname{diag}(d_1^n,d_2^n,...,d_n^n)$$

$$= \operatorname{diag}(a_0 + a_1d_1 + a_2d_1^2 + ... + a_nd_1^n,$$

$$a_0 + a_1d_2 + a_2d_2^2 + ... + a_nd_2^n,$$

$$a_0 + a_1d_3 + a_2d_3^2 + ... + a_nd_3^n$$
:
:
$$a_0 + a_1d_n + a_2d_n^2 + ... + a_nd_n^n) = \operatorname{diag}(f(d_1), f(d_2),... f(d_n))$$

170. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square root of the matrix.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

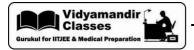
Then
$$A^2 = I$$
 \Rightarrow
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Or
$$\begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & cb + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Or
$$a^2 + bc = 1$$
 (1)

$$ab + bd = 0 (2)$$

$$ac + cd = 0 (3)$$



$$cb + d^2 = 1 (4)$$

If a+d=0, the above four equations hold simultaneously if d=-a and $a^2+bc=1$. Hence, one possible square root of I is $A=\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$

Where $\,\alpha,\beta,\gamma$ are the three numbers related by the condition $\,\alpha^2+\beta\gamma=1$.

If $a + d \neq 0$, then above four equations hold simultaneously

If b = 0, c = 0, a = 1, d = 1 or if b = 0, c = 0, a = -1, d = -1.

Hence, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ i.e., $\pm I$ are other possible square roots of I.

171. First, we will show that I - S is nonsingular. The equality |I - S| = 0 implies that I is a characteristic root of the matrix S, but this is not possible, for a real skew-symmetric matrix can have zero or purely imaginary numbers as its characteristic roots. Thus. $|I - S| \neq 0$, i.e., |I - S| = 0 is nonsingular. We have

$$A^{T} = \left[(I - S)^{-1} \right]^{T} (I + S)^{T} = \left[(I - S)^{T} \right]^{-1} (I + S)^{T}$$

But $(I-S)^{T} = I^{T} - S^{T} = I + S$ and $(I+S)^{T} = I^{T} + S^{T} = I - S$

$$A^{T} = (I + S)^{-1} (I - S)$$

$$A^{T}A = (I - S)^{-1}(1 - S)(I + S)(I - S)^{-1} = (I - S)^{-1}(I + S)(I - S)(I - S)^{-1} = I$$

Thus, A is orthogonal.

172. $(xf)' = xf' + f \text{ and } (x^2 f)'' = [2xf + x^2 f']' = 2f + 4xf' + x^2 f''$

$$\Rightarrow \Delta = \begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ 2f + 4xf' + x^2f'' & 2g + 4xg' + x^2g'' & 2h + 4xh' + x^2h'' \end{vmatrix}$$

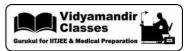
 $R_2 \rightarrow R_2$ - R_1 and then $R_3 \rightarrow R_3$ - $4R_2$ - $2R_1$

$$\Rightarrow \qquad \Delta = \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix}$$

Taking x common from R₂ and multiplying with R₃, we have $\Delta = \begin{bmatrix} f & g & h \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{bmatrix}$

$$\Rightarrow \frac{d\Delta}{dx} = \begin{vmatrix} f' & g' & h' \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$



173. Since α is a repeated root of the quadratic equation f(x) = 0, f(x) can be written as $f(x) = k(x - a)^2$, where k is some nonzero constant.

Let
$$g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$
 $g(x)$ is divisible by $f(x)$ if it is divisible by $(x - \alpha)^2$, i.e., $g(\alpha) = 0$ and

 $g'(\alpha) = 0$.

As A(x), B(x), and C(x) are polynomials of degrees 3, 4 and 5, respectively deg. $g(x) \ge 2$.

Now,
$$g(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$$
 (R₁ and R₂ are identical)

$$\mathsf{Also}\,g'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} \qquad \therefore \quad g'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0 \qquad (R_1 \text{ and } R_3 \text{ are identical}))$$

This implies that f(x) divides g(x).